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LETTER TO THE EDITOR

A Potts model with infinitely degenerate ground state

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Abstract. A q-component Potts model with ferromagnetic and antiferromagnetic interactions in the two respective directions of a square lattice is considered. An argument is given showing that an ordered phase can exist in this model, even though the ground state is disordered and infinitely degenerate. We use the Migdal–Kadanoff transformation to obtain a closed-form expression for its critical point. We also carry out a Monte Carlo simulation of the model for $q = 3$. The specific heat exhibits a broad maximum which does not sharpen appreciably as the lattice size is increased. This suggests that the phase transition, if it exists, is of an unconventional type.

The q-component antiferromagnetic Potts model has generally been neglected in the past, largely due to the expected absence of a long-range order of the usual type for $q \geqslant 3$. The problem has attracted increasing attention recently, after it was suggested (Berker and Kadanoff 1980) that a low-temperature phase of algebraic order may exist in such systems. The central problem of interest has been the possible existence of a transition and, if it exists, the clarification of its nature, in systems whose ground state is infinitely degenerate. For the Potts model with pure antiferromagnetic interactions, it now appears that a transition of some sort exists in three dimensions for $q = 3$ (Berker and Kadanoff 1980, Banavar *et al* 1980). But the situation in dimension $d = 2$ is less clear. Particularly for the $q = 3$ square lattice model a Monte Carlo simulation (Grest and Banavar 1981) indicates a continuous transition at a non-zero temperature, while the exact result of a decorated model (Wu 1981), the rescaling argument (Berker and Kadanoff 1980) and a phenomenological renormalisation group analysis (Schick 1981) imply otherwise. There is a need of further clarification.

In this Letter we consider a two-dimensional Potts model with mixed ferromagnetic and antiferromagnetic interactions. As in the model with pure antiferromagnetic interactions, the model is characterised by an infinitely degenerate ground state. We present an argument showing that, although the ground state is disordered, an ordered phase can still exist in this model at non-zero temperatures. We then determine its critical point in a consistent fashion using the Migdal–Kadanoff bond-moving scheme (Kadanoff 1976). To test the validity of the critical point so obtained and to clarify the nature of the transition we carry out a Monte Carlo simulation of this model for $q = 3$. Our numerical results support the view that, if a transition exists in this model, it is of an unconventional type.

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We consider a Potts model on a square lattice of M columns and N rows. The Hamiltonian $\mathcal{H}$ is given by

$$-\mathcal{H}kT = K_x \sum^{(M)} \delta(\sigma_i, \sigma_j) + K_y \sum^{(N)} \sigma(\sigma_i, \sigma_j) \tag{1}$$

where the summations are over the M and N nearest neighbours in the two respective directions. Here $\sigma_i = 1, 2, \ldots, q$ specifies the spin state at the ith site, $i = 1, 2, \ldots, MN$. We shall consider the case of ferromagnetic $K_x > 0$ and antiferromagnetic $K_y < 0$.

The ground state degeneracy of this system is $q(q-1)^{N-1}$ which becomes infinite in the thermodynamic limit. However, the ground state entropy is not extensive, a feature shared by many other systems with widely varied critical behaviours. Examples are the KDP model of a ferroelectric (Lieb and Wu 1972) which does not exhibit a transition, the hard-square lattice gas with second-neighbour repulsion (Kinzel and Schick 1981) and the domino model (André *et al* 1979, Villain *et al* 1980) for which a transition exists. For the present model we shall argue as in the case of the domino model that an ordered phase can exist at non-zero temperatures, even though the ground state is disordered. It is therefore not unreasonable that a transition exists in this system.

Consider first the case of infinite ferromagnetic interaction $K_x = \infty$. In this limit the system consists of chains in the x direction that are ferromagnetically aligned. The system is then essentially one-dimensional (in the y direction) with an effective coupling MK_y between neighbouring chains.

Such a one-dimensional system is easily solved using the transfer matrix. The transfer matrix has the eigenvalues $\lambda_1 = e^{MK_y} + q - 1$, $\lambda_2 = \lambda_3 = \cdots = \lambda_q = e^{MK_y} - 1$, and as usual, the correlation length ξ is given in terms of the eigenvalues as

$$\xi^{-1} = \ln\left|\frac{\lambda_1}{\lambda_2}\right| = \ln\left|\frac{e^{MK_y} + q - 1}{e^{MK_y} - 1}\right|. \tag{2}$$

It is clear that for M finite ($N \to \infty$) and $K_y < 0$, the zero-temperature correlation length remains finite for $q > 2$,

$$\xi^{-1}(T = 0) = \ln(q - 1). \tag{3}$$

This indicates that the system is disordered at zero temperature. (Notice, however, that $\xi(T = 0)$ diverges if $K_y > 0$.)

Consider next the case of $K_x \neq \infty$. Since the ferromagnetic interactions dominate at low temperatures, to a first approximation we may regard the system as composed of ferromagnetic chains interacting with some effective interactions. This effective interaction can be evaluated as in the consideration of the domino model (Villain *et al* 1980). Consider three neighbouring x-chains labelled A, B and C respectively. The effective interaction between chains A and C is given by summing the Boltzmann weights over all states of chain B. We restrict ourselves to ferromagnetic configurations in chains A and C. This now defines an effective interaction MK'_y between the chains A and C given by

$$\exp(MK'_y) = Z_M(1, 1)/Z_M(1, 2) \tag{4}$$

where $Z_M(a, b)$ is the partition sum of chain B (of length M) with fixed configurations $(a, a, \ldots, a)$ and $(b, b, \ldots, b)$ in chains A and C respectively.

The partition sum $Z_M(a, b)$ is again most conveniently carried out using the transfer matrix. After some straightforward algebra the result yields, in the limit of $M \to \infty$,

$$\exp(K'_y) = (u + uv^2 + q - 2 + \sqrt{E})/(u + uv + q - 3 + v + \sqrt{F}) \tag{5}$$

where

$$E = u^2(1-v^2)^2 + 2(q-2)(1-v^2)u + (q-2)^2 + 4(q-1)v^2,$$
$$F = u^2(1-v)^2 + 2(1-v)(q-3-v)u + (q-3)^2 + (6q-10)v + v^2, \qquad (6)$$
$$u = e^{K_x}, \qquad v = e^{K_y}.$$

For $T \to 0$ and $q \neq 2$ this reduces to

$$K_y' = e^{-K_x} + \text{higher-order terms}. \qquad (7)$$

Thus the effective coupling K_y' is zero at $T = 0$ and is ferromagnetic at higher temperatures. Therefore, as in the case of the domino model, it is possible to have an ordered phase at $T \neq 0$. This order is characterised by ferromagnetic correlation between x-chains separated by an even number of lattice spacings in the y (antiferromagnetic) direction.

Assuming the existence of a transition in the present model, we now show that the Migdal–Kadanoff transformation (Kadanoff 1976) can be used in a consistent fashion to determine its critical point.

The Migdal–Kadanoff transformation consists of a sequence of bond-moving and site-decimation processes. We first move sequences of n neighbouring horizontal bonds to form new ones of strength K_x^*. This is followed immediately by a decimation of the sites now without horizontal interactions, leading to new vertical interactions K_y^*. An example of this scheme for $n = 2$ is shown in figure 1.

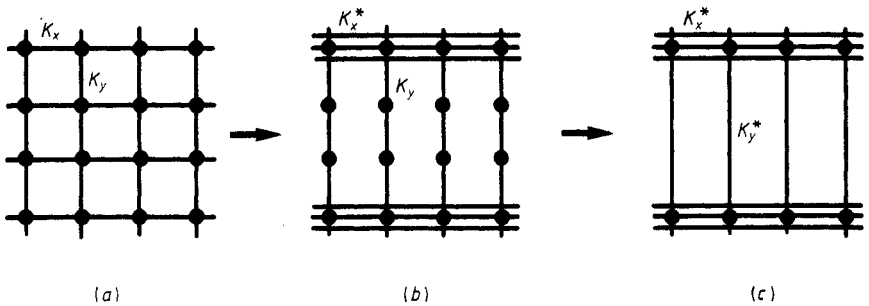

Figure 1. The Migdal–Kadanoff transformation. (*a*) The original lattice. (*b*) Sequences of $n = 2$ horizontal bands are moved. (*c*) Sites with no horizontal interactions are decimated.

It is clear that

$$K_x^* = (n+1)K_x > 0 \qquad (8)$$

and also we have after the site decimation (see e.g. Kinzel and Domany 1981)

$$t(K_y^*) = [t(K_y)]^{n+1} \qquad (9)$$

where

$$t(K) = 1 + q/(e^K - 1). \qquad (10)$$

It follows that, for $q > 0$, we have $K_y^* > 0$ provided that we take $n = \text{odd}$. Now for ferromagnetic $K_x^* > 0$ and $K_y^* > 0$ the critical point of the Potts model is (see e.g. Wu 1981)

$$\exp(K_x^*) = t(K_y^*). \qquad (11)$$

Upon using (8) and (9), (11) leads to the following critical condition for the original lattice:

$$(1+e^{K_x})(1-e^{K_y})=q \qquad (K_x>0, K_y<0). \tag{12}$$

The validity of (12) can be verified in one special case. For $q=2$ the critical point of the model is known; it is readily verified that for $q=2$ (12) agrees with the exact result $e^{K_x}=t(-K_y)$. The relation (12) is also self-consistent in the sense that, if one chooses n = even in the Migdal–Kadanoff scheme so as to arrive at $K_x^*>0$ and $K_y^*<0$, (12) is also satisfied by these new couplings. Note that this invariance property is not expected *a priori*, as it breaks down if one attempts to generalise the consideration to models with pure antiferromagnetic interactions.

We next carry out a Monte Carlo simulation of the model (1) for $q=3$. It has been recently shown by Ostlund (1981) that the present model for $q=3$ and $K_x=-K_y$ is a special case of the three-state asymmetric clock model, and on the basis of the free-fermion approximation of Villain and Bak (1981), Ostlund (1981) has reached the conclusion that an XY-type transition occurs in this model at a non-zero temperature. It will therefore be very illuminating to examine our model numerically.

We used the standard Monte Carlo technique (Binder 1979), computing the energy and the specific heat for systems of sizes $N\times N$ with periodic boundary conditions, where N ranges from 4 to 40. Averages were taken over a few thousand Monte Carlo steps per site (MCS/S) for the large systems and about 10^4 MCS/S for $N=4$.†.

Our main findings are summarised in figure 2 where the specific heat maxima are plotted as a function of the ratio $-K_y/K_x$. We also plot the critical condition (12) determined from the Migdal–Kadanoff transformation for comparison. In our simulations the locations of the specific heat maxima follow closely (12) for $N=20$. However,

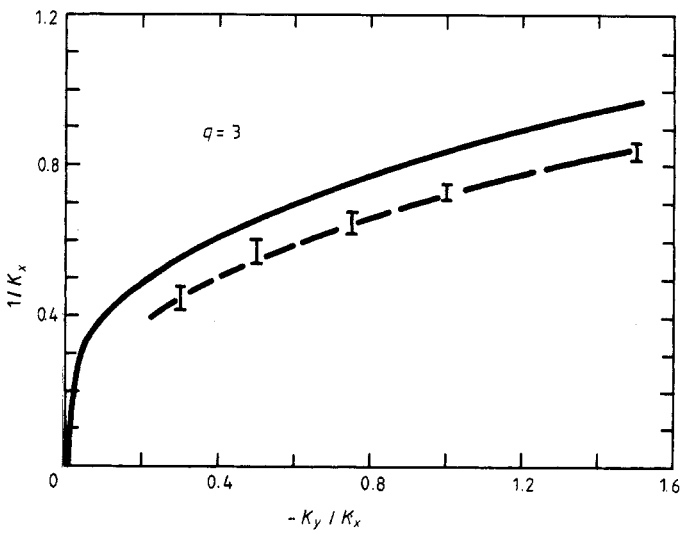

Figure 2. Critical temperature for the $q=3$ model. The full curve depicts (12). The error bars show the Monte Carlo results for the location of the maxima in the specific heat as systems of sizes 20×20.

† When applied to the ferromagnetic system $K_x=K_y>0$, our procedure leads to data in agreement with those of Binder (1981).

a study of the size dependence for the system $(K_x, K_y) \sim (1, -1)$ shows that the specific heat maximum moves to lower temperatures for larger systems, leading to a distinct deviation from the critical condition indicated by (12). This size dependence is shown in figure 3. More significantly, the specific heat peak sharpens only slightly with a very small increase in its height as N is raised appreciably. This behaviour is reminiscent of that found in the Kosterlitz–Thouless, XY-type transitions (van Himbergen and Chakravarty 1981, Selke 1981), and is in sharp contrast with that of the ferromagnetic Potts model. In figure 3 we show also the size dependence for a ferromagnetic model with $(K_x, K_y) \sim (2, 0.2072)$. (These couplings are chosen to correspond to those parameters obtained after performing an $n = 1$ Migdal–Kadanoff transformation on $(K_x, K_y) \sim (1, -1)$.) It is clear that the size dependence in the ferromagnetic model indicates a strong divergence in the specific heat, which is absent in the model (1).

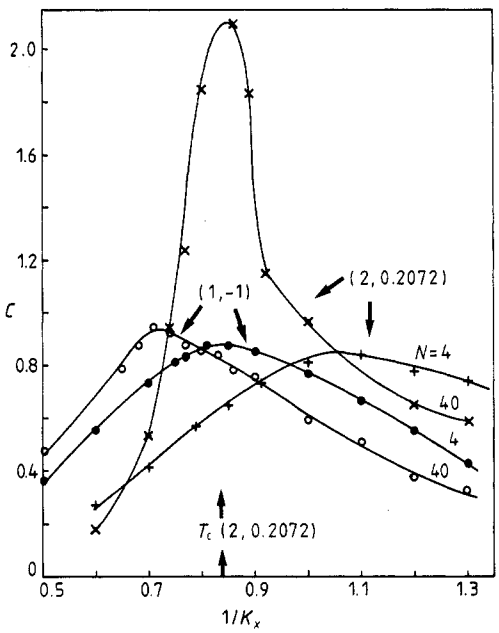

Figure 3. Temperature and size dependences of the specific heat, C (in units of the universal gas constant), for the model $(K_x, K_y) \sim (1, -1)$ and the 'corresponding' ferromagnetic model with $(K_x, K_y) \sim (2, 0.2072)$. T_c denotes the exact transition temperature of the latter case in the thermodynamic limit.

We interpret the Monte Carlo findings as suggesting that the transition, if it exists, is of an unconventional type. Note that the specific heat maximum is located at a temperature somewhat lower than that given by (12). However, it is not entirely certain that the maximum should be identified as the transition point. In the case of the F model of an antiferroelectric which exhibits an unconventional type infinite-order transition (Lieb 1967), it is known that the specific heat does not diverge and that, as found in the present case, the specific heat maximum occurs at a temperature lower than the exact critical point (Lieb and Wu 1972). Therefore our numerical data do not necessarily rule out the possibility that (12) may very well be exact.

Further work on this model, including an application to a mocked axial next-nearest-neighbour model, is in progress and will be reported in the future.

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